Dynamical sys.

- To study evolution of "something" in time
  $\underbrace{\text{"something"}}_{\text{system}}$

- Sys. is represented by state

$$x = (x_1, x_2, \ldots, x_n) \in \mathcal{X}$$

set of all possible values of x

a) $\mathcal{X}$ is discrete e.g. $\{0, 1, 2, \ldots\}$

b) $\mathcal{X}$ is continuous e.g $\mathbb{R}^n$

- State evolves according to an update law

Discrete-time: $x(k+1) = T(x(k))$ iterated maps

Continous-time: $\dot{x}_{(t)} = f(x_{(t)})$ ODE

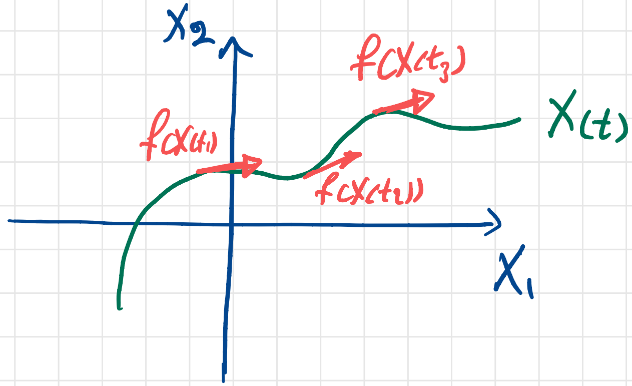
we study this

- $\dot{x} = f(x)$ is compact notation for

$$\frac{dx_1}{dt} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

- $f: \mathbb{R}^n \to \mathbb{R}^n$ is a vector field (comes from physics/simulation)

- vector field $f(\cdot)$ is tangent to the trajectory $x(t)$



- Initial condition

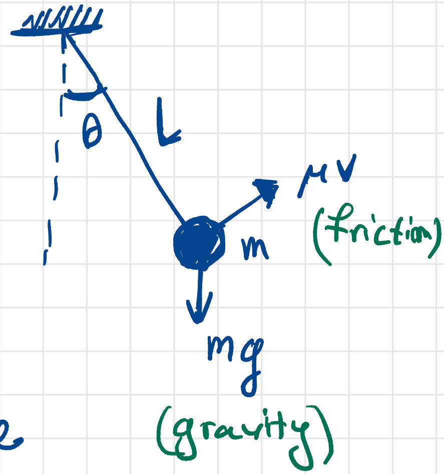
$$\dot{x} = f(x) \qquad x(0) = x_0$$

- Black box view

$$x_0 \longrightarrow \boxed{\dot{x} = f(x)} \longrightarrow x(t)$$

- Underlying principle for dyn. sys.:
  - Future of the sys. only depends on current state
  - $x(t)$ for $t \geq \tau$ only depends on $x(\tau)$
  - Therefore, state encapsulate memory of the sys.
  - This is called Markov property for stochastic sys.

- Example: Pendulum



Q: what is the state?

Q: what is the dyn?

Newton's law: $I\ddot{\theta} = \text{torque}$

Inertia: $I = mL^2$

torque from gravity: $-mgL\sin\theta$

torque from friction: $-\mu L^2 \dot{\theta}$ $\quad (v = L\dot{\theta})$

$$\Longrightarrow \quad mL^2\ddot{\theta} = -mgL\sin\theta - \mu L^2 \dot{\theta}$$

$$\Longrightarrow \quad \ddot{\theta} = -\frac{g}{L}\sin\theta - \underbrace{\frac{\mu}{m}}_{\gamma}\dot{\theta}$$

state $\quad X = (X_1, X_2) = (\theta, \dot{\theta})$

$$\Longrightarrow \quad \dot{X}_1 = \dot{\theta} = X_2$$

$$\dot{X}_2 = \ddot{\theta} = -\frac{g}{L}\sin(X_1) - \gamma X_2$$

$$\Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{L}\sin(x_1) - \gamma x_2 \end{bmatrix}$$

- Equilibibirium point.

x^* is an eql. point $\Longleftrightarrow$ if $x(0) = x^*$ then $x(t) = x^*$

$\Updownarrow$

$f(x^*) = 0$

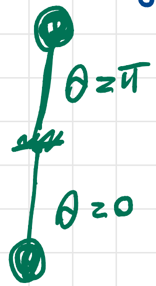
- Example: Pendulum

$f_1(x_1, x_2) = 0 \Longrightarrow x_2 = 0$

$f_2(x_1, x_2) = 0 \Longleftrightarrow \sin(x_1) = 0 \Longrightarrow x_1 = n\pi$ for integer n

there are only two physically distinguishable solution

$x_1 = 0$ or $x_1 = \pi$



Controlled Dynamical system:

- Dyn. sys. driven by control input

$$\dot{x} = f(x,u), \qquad x(0) = x_0$$

- $u = (u_1, u_2, \ldots, u_p) \in \mathbb{R}^p$ is the control input

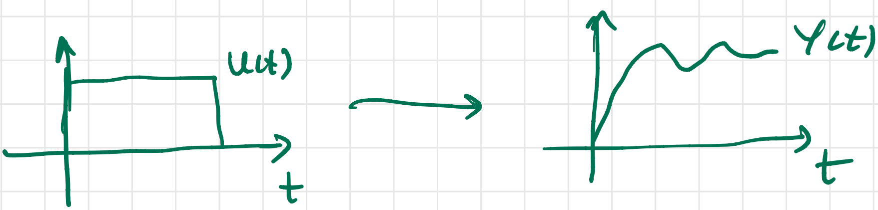
- Output signal or observation

$$y = h(x,u)$$

e.g. IMU sensors

- Black-box or input-output viewpoint

$$x_0, u(t) \longrightarrow \boxed{\begin{array}{l}\dot{x} = f(x,u)\\ y = h(x,u)\end{array}} \longrightarrow y(t)$$



Outline of the class

Part I: $\dot{x} = f(x)$ fundamental of ODE stability, ...
(up to midterm)

part II: $\dot{x} = f(x, u)$ control design
$y = h(x, u)$

Main tool: Lyapunov function. method

- We do not study time-varying systems
- time varying sys. can be reduced to time invariant

$$\dot{x} = f(x, t)$$

define new state $x_{n+1} = t$, with dyn. $\dot{x}_{n+1} = 1$

Review of linear sys.

$$\dot{x} = Ax, \qquad x(0) = x_0$$

- A is a $n \times n$ matrix
- Solution is explicitly known

$$x(t) = e^{tA} x_0$$

- $x^* = 0$ is eqlb point.
- Behavior of linear sys. depends on spectrum of A.

superposition

- why are linear sys. called linear?

if

$$x^1(0) \longrightarrow \boxed{\dot{x} = Ax} \longrightarrow x^1(t)$$

$$x^2(0) \longrightarrow \boxed{\dot{x} = Ax} \longrightarrow x^2(t)$$

then

$$a x^1(0) + b x^2(0) \longrightarrow \boxed{\dot{x} = Ax} \longrightarrow a x^1(t) + b x^2(t)$$

- linear sys. with control input

$$\dot{X} = AX + Bu \quad , \quad X_0 = 0$$
$$Y = CX + Du$$

usual assumption

- Input-output relationship is characterized by transfer function

Laplace-transform: $X(t) \longrightarrow \hat{X}(s) = \int_0^\infty e^{-st} X(t)\,dt$

$$\Longrightarrow \quad s\hat{X}(s) = A\hat{X}(s) + B\hat{u}(s)$$
$$\hat{Y}(s) = C\hat{X}(s) + D\hat{u}(s)$$

$$\Longrightarrow \quad \hat{X}(s) = (sI - A)^{-1} B \hat{u}(s)$$

$$\Longrightarrow \quad \hat{Y}(s) = \underbrace{\left[C(sI-A)^{-1}B + D \right]}_{G(s)} \hat{u}(s)$$

transfer function

- Superposition holds

if

$$u^{(1)}(t) \longrightarrow \boxed{\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}} \longrightarrow y^{(1)}(t)$$

$$u^{(2)}(t) \longrightarrow \boxed{\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}} \longrightarrow y^{(2)}(t)$$

Then

$$a u^{(1)}(t) + b u^{(2)}(t) \longrightarrow \boxed{\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}} \longrightarrow a y^{(1)}(t) + b y^{(2)}(t)$$

- When superposition does not hold $\rightarrow$ <u>Non linear</u>

- Results in interesting phenomenons.

① Multiple eqlb. points

$\downarrow$

we saw this in Pendulum example

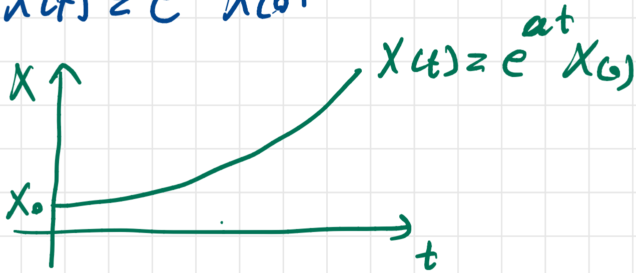
② Finite-time blow-up

a) linear sys. can not go to ∞ in finite-time

e.g. $x \in \mathbb{R}$, $\dot{x} = a x$ for $a > 0$

then $x(t) = e^{at} x(0)$

It is always finite even for very large a



b) example of nonlinear sys.

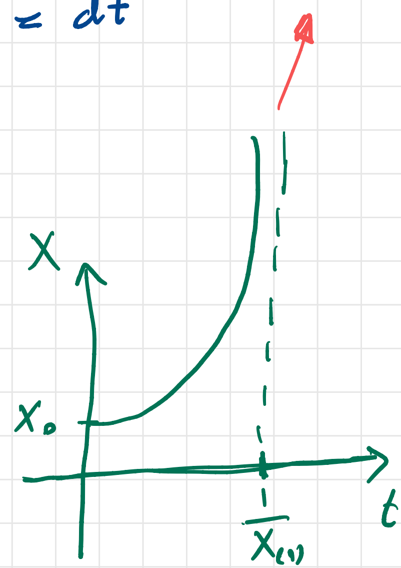
$$\dot{x} = x^2 \Rightarrow \frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt$$

$$\Rightarrow \int_{x(0)}^{x(T)} \frac{dx}{x^2} = \int_0^T dt$$

$$\Rightarrow -\frac{1}{x(T)} + \frac{1}{x(0)} = T$$

$$\Rightarrow x(T) = \frac{x(0)}{1 - T x(0)}$$

finite time blow up



③ Sub/super harmonic oscillations

- For linear sys. if input has freq. ω then the output has the same freq:

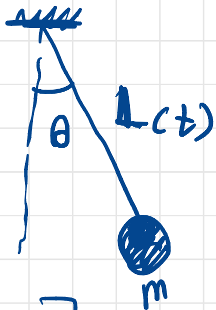
$$u(t) = \cos(\omega t) \implies y(t) = A\cos(\omega t + \phi)$$

Bode plots

- Not true for nonlinear sys.

- Example: Pumping up a swing

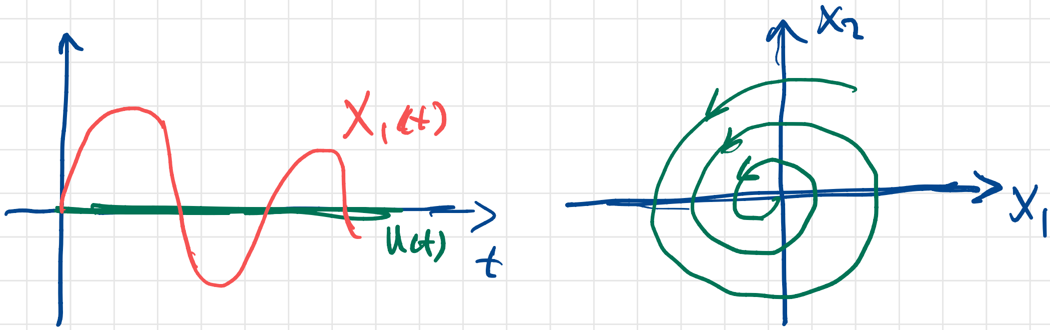
Consider the Pendulum, but length can be Controlled.



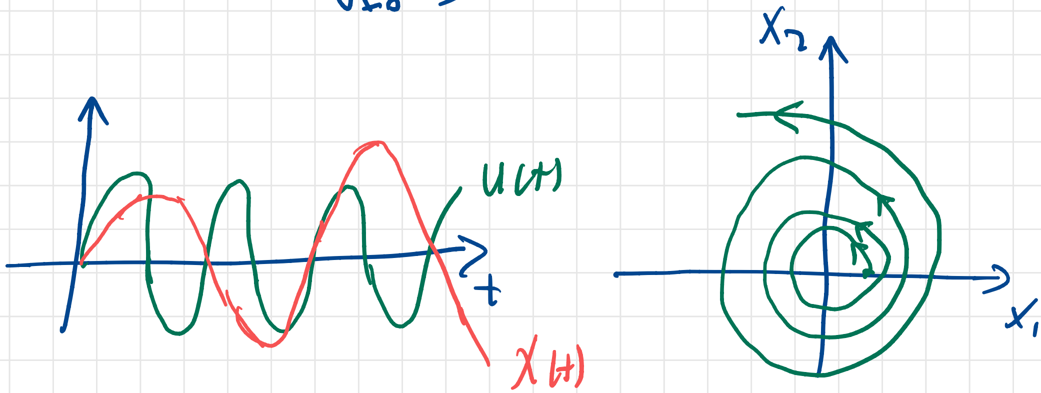
$$L(t) = L_0(1 + \varepsilon \underbrace{u(t)}_{\text{Control input}})$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{L_0(1+\varepsilon u(t))} \sin(x_1) - \gamma x_2 \end{bmatrix}$$

a) $u(t) = 0$



b) $u(t) = \cos\left(2\sqrt{\frac{g}{l_0}}\, t\right)$



Ref: Pumping a swing, by Piccoli & Kulkarni, CSM

Related to Mathieu eq.

Other nonlinear phenomenons:

- limit cycles
- Bifurcation
- Chaos

important, but not in the scope of this class

Read Ch. 1